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ON HEILBRONN'S PROBLEM IN HIGHER DIMENSION

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Heilbronn conjectured that given arbitrary n points in the 2-dimensional unit square $[0,1]^2$, there must be three points which form a triangle of area at most $O(1/n^2)$. This conjecture was disproved by a nonconstructive argument of Komlós, Pintz and Szemerédi [10] who showed that for every n there is a configuration of n points in the unit square $[0,1]^2$ where all triangles have area at least $\Omega(\log n/n^2)$. Considering a generalization of this problem to dimensions $d \geq 3$, Barequet [3] showed for every n the existence of n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum volume of every simplex spanned by any $(d+1)$ of these n points is at least $\Omega(1/n^d)$. We improve on this lower bound by a logarithmic factor $\Theta(\log n)$.

1. Introduction

An old conjecture of Heilbronn states that for every distribution of n points in the 2-dimensional unit square $[0,1]^2$ (or unit disc) there are three distinct points which form a triangle of area at most c/n^2 for some constant $c>0$. Erdős observed that this conjecture, if true, would be best possible, as, for n a prime, the points $(i, i^2 \bmod n)_{i=0,\ldots,n-1}$ in the $n \times n$ grid would show after rescaling, see [2]. However, Komlós, Pintz and Szemerédi [10] disproved Heilbronn's conjecture by showing for every n the existence of a configuration of n points in $[0,1]^2$ with every three of these n points forming a triangle of area at least $c' \cdot \log n/n^2$ for some constant $c'>0$. This existence argument was made constructive in [5], where a deterministic polynomial time algorithm was given, which finds n points in $[0,1]^2$ achieving this lower bound

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$\Omega(\log n/n^2)$ on the minimum triangle area. Upper bounds on Heilbronn's triangle problem were given by Roth in a series [12–16] of papers and by Schmidt [18], see Rothschild and Straus [17] for related results, and the currently best upper bound $O(n^{-8/7+\varepsilon})$ for every fixed $\varepsilon > 0$ is due to Komlós, Pintz and Szemerédi [9].

Recently, Barequet [3] considered a d-dimensional version of Heilbronn's problem. For given $(d+1)$ vectors $p_1, \ldots, p_{d+1} \in \mathbb{R}^d$ the set $\{\sum_{i=1}^{d+1} \lambda_i \cdot p_i \mid \sum_{i=1}^{d+1} \lambda_i = 1;\ \lambda_1, \ldots, \lambda_{d+1} \geq 0\}$ is called a *simplex*. For fixed dimension $d \geq 3$, Barequet showed for every n, that there exist n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum volume of every simplex spanned by any $(d+1)$ of these n points is at least $\Omega(1/n^d)$. He gave three different approaches towards a solution of the problem. The first one, for dimension $d = 3$, uses a Greedy-type argument, i.e., adding to given points a new point as long as possible, such that no two points are too close, no three points form a triangle of too small area and no four points form a tetrahedron of too small volume (see also [18] for the case $d=2$). With this he obtained a configuration of n points in the 3-dimensional unit cube $[0,1]^3$ such that the minimum volume of every tetrahedron is at least $\Omega(1/n^4)$. The second approach, which yields a better lower bound, was worked out for every fixed dimension $d \geq 3$ and uses a random argument: if $2n$ points are dropped uniformly at random and independently of each other in the d-dimensional unit cube $[0,1]^d$, then the expected number of simplices with volume at most c_d/n^d is at most n, where $c_d > 0$ is a constant. Deleting one point from every such small simplex yields the existence of n points in $[0,1]^d$ with every simplex formed by $(d+1)$ of these points having volume at least $\Omega(1/n^d)$. The third approach however is similar to Erdős one's (and according to Bollobás [6] was known to him) and is an explicit construction, namely taking the points $P_k = 1/n \cdot (k^j \bmod n)_{j=1,\ldots,d}$ for $k = 0, 1, \ldots, n-1$ on the moment curve. The volume of every simplex is given by the determinant of a Vandermonde matrix, which is not equal to 0 for n a prime, multiplied by $\Theta(1/n^d)$ and this gives minimum value at least $\Omega(1/n^d)$.

Note, that the corresponding problem in dimension $d=1$ is trivial as n equidistant points in the unit interval $[0,1]$ show.

Here we will improve Barequet's lower bound for dimensions $d \geq 3$, using a probabilistic existence argument, by a factor $\Theta(\log n)$:

Theorem 1.1. *For every fixed integer $d \geq 2$ and for every n there exists a configuration of n points in the unit cube $[0,1]^d$ such that the volume of every simplex spanned by any $(d+1)$ of these points is at least $\Omega(\log n/n^d)$.*

2. Simplices with Small Volume and Hypergraphs

In our arguments we will use hypergraphs. The parameters *independence number* of a hypergraph and 2-*cycles* will be important in our considerations:

Definition 2.1. Let $\mathcal{G}=(V,\mathcal{E})$ be a *hypergraph* with vertex set V and edge set $\mathcal{E}$ where each edge $E\in\mathcal{E}$ satisfies $E\subseteq V$. A hypergraph $\mathcal{G}=(V,\mathcal{E})$ is k-*uniform* if every edge $E\in\mathcal{E}$ contains exactly k vertices.

A subset $I\subseteq V$ is called *independent* if I contains no edge $E\in\mathcal{E}$. The largest size of an independent set in $\mathcal{G}$ is called the *independence number* $\alpha(\mathcal{G})$.

In a k-uniform hypergraph $\mathcal{G}=(V,\mathcal{E})$, $k\geq 3$, a 2-*cycle* is a pair $\{E_1,E_2\}$ of distinct edges $E_1,E_2\in\mathcal{E}$ with $|E_1\cap E_2|\geq 2$. A 2-cycle $\{E_1,E_2\}$ in $\mathcal{G}$ is called $(2,j)$-*cycle* if $|E_1\cap E_2|=j$, where $j=2,\dots,k-1$.

We will reformulate the geometrical problem considered by Barequet as a problem of finding in an appropriately defined hypergraph a large independent set. For a given set $S\subseteq[0,1]^d$ of points we form a $(d+1)$-uniform hypergraph with vertex set being this set S of points in $[0,1]^d$. The edges are determined by all subsets of $(d+1)$ points from S, which form a simplex of 'small' volume, to be specified later. An independent set in this hypergraph corresponds to a set of points in $[0,1]^d$, where no simplex has 'small' volume. In order to show the existence of a large independent set, we will use the following result of Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], stated here in a variant proven in [7]:

Theorem 2.2 **([1],[7]).** *Let $k\geq 3$ be a fixed integer. Let $\mathcal{G}=(V,\mathcal{E})$ be a k-uniform hypergraph on $|V|=n$ vertices and with average degree $t^{k-1}=k\cdot|\mathcal{E}|/n$. If $\mathcal{G}$ does not contain any 2-cycles, then the independence number $\alpha(\mathcal{G})$ satisfies for some constant $c_k>0$:*

$$\alpha(\mathcal{G})\geq c_k\cdot\frac{n}{t}\cdot(\log t)^{\frac{1}{k-1}}\,.$$

In recent years, several applications and also an algorithmic version of Theorem 2.2 have been found, compare [4]. Here we will give a another application of this deep result.

In d dimensions the volume of a simplex determined by the points $P_1,\dots,P_{d+1}\in[0,1]^d$ is given by $\mathrm{vol}(P_1,\dots,P_{d+1}):=1/d\cdot G\cdot h$, where G is the volume of the simplex determined by the points $P_1,\dots,P_d$ (in the corresponding $(d$–$1)$-dimensional subspace) and h is the Euclidean distance of the point P_{d+1} from the hyperplane given by $P_1,\dots,P_d$. Thus, if h_k denotes the

Euclidean distance of P_k from the hyperplane determined by $P_1,\dots,P_{k-1}$, $k=2,\dots,d+1$, then

$$\text{vol}\,(P_1,\dots,P_{d+1}) = \frac{1}{d!}\cdot\prod_{k=2}^{d+1} h_k\,.$$

In the following we will prove Theorem 1.1.

Proof. In the d-dimensional unit cube $[0,1]^d$ we drop $n^{1+\varepsilon}$ points uniformly at random and independently of each other, where ε is a small constant with $0<\varepsilon<1/(2d)$. On this random set of points $P_1,\dots,P_{n^{1+\varepsilon}}$ we form a random $(d+1)$-uniform hypergraph $\mathcal{G}(\beta)=(V,\mathcal{E})$ with the vertices being the $n^{1+\varepsilon}$ random points in $[0,1]^d$, thus $|V|=n^{1+\varepsilon}$. Every $(d+1)$ vertices $P_{i_1},\dots,P_{i_{d+1}}$, of these $n^{1+\varepsilon}$ vertices form an edge in $\mathcal{G}(\beta)$ if the volume $\text{vol}\,(P_{i_1},\dots,P_{i_{d+1}})$ of the corresponding simplex is at most β, i.e., $\{P_{i_1},\dots,P_{i_{d+1}}\}\in\mathcal{E}$ if and only if $\text{vol}\,(P_{i_1},\dots,P_{i_{d+1}})\le\beta$. We will show for the choice $\beta:=c\cdot\log n/n^d$, where $c>0$ is a suitable constant, that among these $n^{1+\varepsilon}$ vertices there exists an independent set of n vertices. Then, every simplex determined by $(d+1)$ distinct points of these n points has volume at least $\Omega(\log n/n^d)$.

First we estimate the expected number $E(|\mathcal{E}|)$ of edges in the random hypergraph $\mathcal{G}(\beta)$.

Lemma 2.3. *For some constant $C_d>0$, the expected number $E(|\mathcal{E}|)$ of edges in the random hypergraph $\mathcal{G}(\beta)=(V,\mathcal{E})$ satisfies:*

$$E(|\mathcal{E}|)\le C_d\cdot\beta\cdot n^{(1+\varepsilon)(d+1)}\,. \tag{1}$$

Proof. Our arguments are similar to those in [3]. We give an upper bound on the probability $\text{Prob}\,(\text{vol}\,(P_1,\dots,P_{d+1})\le\beta)$ that $(d+1)$ points $P_1,\dots,P_{d+1}$ dropped in $[0,1]^d$, uniformly at random and independently of each other, form a simplex of volume at most β, i.e., we will show for some constant $C'_d>0$ and for every $\beta>0$:

$$\text{Prob}\,(\text{vol}\,(P_1,\dots,P_{d+1})\le\beta)\le C'_d\cdot\beta\,. \tag{2}$$

For $k=2,\dots,d+1$, let x_k denote the Euclidean distance of P_k from the $(k-2)$-dimensional hyperplane H_{k-1} determined by the points $P_1,\dots,P_{k-1}$. Assume that the points $P_1,\dots,P_{k-1}$, $k=2,\dots,d$, are already fixed. We estimate the probability that the Euclidean distance x_k lies in the infinitesimal range $[g_k,g_k+dg_k]$. Taking the differences of the corresponding volumes of the cylinders determined by all points with Euclidean distance at most (g_k+dg_k) and g_k, respectively, (which are given by the volumes of $(d+2-k)$-dimensional balls with radii (g_k+dg_k) and g_k, respectively, multiplied by some positive constant, which depends on d only) from the hyperplane H_{k-1}, we infer for some constant $c_d > 0$:

$$\text{Prob}\,(g_k \le x_k \le g_k + dg_k) \le d(c_d \cdot g_k^{d+2-k}) = c_d \cdot (d+2-k) \cdot g_k^{d+1-k} dg_k \,.$$

Now, having fixed the points $P_1, \ldots, P_d$, the point P_{d+1} must fulfill $\text{vol}\,(P_1, \ldots, P_{d+1}) \le \beta$, hence the Euclidean distance x_{d+1} of P_{d+1} from the hyperplane determined by $P_1, \ldots, P_d$ must satisfy

$$\frac{1}{d!} \cdot x_{d+1} \cdot \prod_{k=2}^{d} g_k \le \beta \,.$$

The Euclidean distance between two points in $[0,1]^d$ is at most $\sqrt{d}$, thus, the point P_{d+1} must lie within a box of base area at most $(\sqrt{d})^{d-1}$ and of height at most

$$2 \cdot d! \cdot \frac{\beta}{\prod_{k=2}^{d} g_k} \,,$$

which happens with probability at most

$$2 \cdot d! \cdot (\sqrt{d})^{d-1} \cdot \frac{\beta}{\prod_{k=2}^{d} g_k} \,.$$

The distances $x_2, \ldots, x_d$ can be arbitrary within the range $[0, \sqrt{d}]$. Collecting constant factors, which only depend on the dimension d to constants $C_d', C_d'' > 0$, we infer

$$\begin{aligned}
&\text{Prob}\,(\text{vol}\,(P_1, \ldots, P_{d+1}) \le \beta) \\
&\quad \le \int_0^{\sqrt{d}} \cdots \int_0^{\sqrt{d}} \left(\prod_{k=2}^{d} c_d \cdot (d+2-k) \cdot g_k^{d+1-k} \right) \\
&\qquad\qquad \cdot \frac{2 \cdot d! \cdot (\sqrt{d})^{d-1} \cdot \beta}{\prod_{k=2}^{d} g_k} \, dg_d \ldots dg_2 \\
&\quad = C_d'' \cdot \beta \cdot \int_0^{\sqrt{d}} \cdots \int_0^{\sqrt{d}} \prod_{k=2}^{d} g_k^{d-k} \, dg_d \ldots dg_2 \\
&\quad = C_d'' \cdot \beta \cdot \frac{1}{(d-1)!} \cdot d^{\frac{d \cdot (d-1)}{4}} \\
&\quad = C_d' \cdot \beta \,.
\end{aligned}$$

There are $\binom{n^{1+\varepsilon}}{d+1}$ possibilities to choose $(d+1)$ out of the $n^{1+\varepsilon}$ random points, hence with (2) for some constant $C_d > 0$ the expected number $E(|\mathcal{E}|)$

of edges in the random hypergraph $\mathcal{G}(\beta)=(V,\mathcal{E})$ satisfies:

$$E(|\mathcal{E}|) \le C'_d \cdot \beta \cdot \binom{n^{1+\varepsilon}}{d+1} \le C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)} \, . \qquad \blacksquare$$

To apply Theorem 2.2, we will show that the expected number of *'bad configurations'* among the $n^{1+\varepsilon}$ random points is small, i.e., much less than $n^{1+\varepsilon}$. These bad configurations are pairs of points with small Euclidean distance and 2-cycles in the hypergraph $\mathcal{G}(\beta)$.

First we give an upper bound on the probability that there exist two distinct points P, Q among the $n^{1+\varepsilon}$ random points which have Euclidean distance $\text{dist}\,(P,Q)$ less than some value $D>0$.

Lemma 2.4. *For every real number $D>0$ and random points $P_1,\ldots,P_{n^{1+\varepsilon}} \in [0,1]^d$ it is*

(3) $$\text{Prob}\,(\exists k \ne l : \text{dist}\,(P_k,P_l) < D) \le c_d \cdot D^d \cdot n^{2+2\varepsilon} \, .$$

Proof. For a fixed point P_k, the probability that the point P_l, $l \ne k$, has Euclidean distance less than D from P_k, is given by the volume of the d-dimensional ball with center P_k and radius D, i.e., by $c'_d \cdot D^d$ for some constant $c'_d > 0$. Since there are $\binom{n^{1+\varepsilon}}{2}$ choices for the points P_k and P_l, we have for some constant $c_d > 0$:

$$\begin{aligned} &\text{Prob}\,(\exists k \ne l : \text{dist}\,(P_k,P_l) < D) \\ \le &\sum_{1 \le k < l \le n^{1+\varepsilon}} \text{Prob}\,(\text{dist}\,(P_k,P_l) < D) \\ \le &\binom{n^{1+\varepsilon}}{2} \cdot c'_d \cdot D^d \le c_d \cdot D^d \cdot n^{2+2\varepsilon} \, . \qquad \blacksquare \end{aligned}$$

With (3) and $D_0 := n^{-2/(d-1)}$, where $0<\varepsilon<2/(d-1)$, we obtain that

$$\text{Prob}\,(\exists k \ne l : \text{dist}\,(P_k,P_l) < D_0) = o(1) \, ,$$

thus,

(4) $$\text{Prob}\,(\forall k \ne l : \text{dist}\,(P_k,P_l) \ge D_0) = 1 - o(1) \, ,$$

and with probability close to 1 distinct points have Euclidean distance at least D_0.

Next, for $j = 2,\ldots,d$, we will give an upper bound on the conditional expected numbers

$$E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \ne l : \text{dist}\,(P_k,P_l) \ge D_0)$$

of $(2,j)$-cycles in $\mathcal{G}(\beta)$, that is, the expected numbers of pairs $\{E_1,E_2\}$ of edges $E_1,E_2\in\mathcal{E}$ with $|E_1\cap E_2|=j$, given that distinct points have Euclidean distance at least D_0.

Lemma 2.5. *For $j=2,\ldots,d-1$ and constants $c_j(d)>0$ the random hypergraph $\mathcal{G}(\beta)$ satisfies:*

$$E(s_{2,j}(\mathcal{G}(\beta))\,|\,\forall k\neq l:\mathrm{dist}\,(P_k,P_l)\geq D_0)\leq c_j(d)\cdot\beta^2\cdot n^{(1+\varepsilon)(2d+2-j)}\,, \tag{5}$$

and for $j=d$ and a constant $c(d)>0$ it is

$$E(s_{2,d}(\mathcal{G}(\beta))\,|\,\forall k\neq l:\mathrm{dist}\,(P_k,P_l)\geq D_0)\leq c(d)\cdot\beta^2\cdot n^{(1+\varepsilon)(d+2)}\cdot\log n\,. \tag{6}$$

Proof. Let $j=2,\ldots,d$. Consider $(2d+2-j)$ random points $P_1,\ldots,P_{2d+2-j}\in[0,1]^d$ where the Euclidean distances satisfy $\mathrm{dist}\,(P_k,P_l)\geq D_0=n^{-2/(d-1)}$ for $1\leq k<l\leq 2d+2-j$. We will give an upper bound on the following conditional probability:

$$\mathrm{Prob}(P_1,\ldots,P_{2d+2-j}\ \text{form a}\ (2,j)\text{-cycle in}\ \mathcal{G}(\beta)\ |\ \forall k\neq l:\mathrm{dist}\,(P_k,P_l)\geq D_0).$$

Let us assume that the two simplices, which yield a $(2,j)$-cycle, are $E=\{P_1,\ldots,P_{d+1}\}\in\mathcal{E}$ and $E'=\{P_1,\ldots,P_j,P_{d+2},P_{d+3},\ldots,P_{2d+2-j}\}\in\mathcal{E}$ with

$$\mathrm{vol}\,(P_1,\ldots,P_{d+1})\leq\beta \tag{7}$$

and

$$\mathrm{vol}\,(P_1,\ldots,P_j,P_{d+2},\ldots,P_{2d+2-j})\leq\beta\,. \tag{8}$$

All possibilities for forming a $(2,j)$-cycle will be taken into account by the constant factor $\binom{2d+2-j}{d+1}\cdot\binom{d+1}{j}$. Let $\mathcal{F}_{E,E'}$ denote the event "$\{E,E'\}$ is a $(2,j)$-cycle in $\mathcal{G}(\beta)$ given that $\forall k\neq l:\mathrm{dist}\,(P_k,P_l)\geq D_0$". We will estimate the probability $\mathrm{Prob}\,(\mathcal{F}_{E,E'})$.

For $k=2,\ldots,d+1$, let x_k denote the Euclidean distance of the point P_k from the hyperplane determined by $P_1,\ldots,P_{k-1}$. For $l=d+2,\ldots,2d+2-j$, let y_l be the Euclidean distance of the point P_l from the hyperplane determined by $P_1,\ldots,P_j,P_{d+2},\ldots,P_{l-1}$, where for $l=d+2$ the hyperplane is determined by $P_1,\ldots,P_j$. Assume that the points $P_1,\ldots,P_{k-1}$, are already fixed. As in the proof of Lemma 2.3 we have for some constant $c'_d>0$:

$$\mathrm{Prob}\,(g_k\leq x_k\leq g_k+dg_k)\leq d(c_d\cdot g_k^{d+2-k})\leq c'_d\cdot g_k^{d+1-k}dg_k\,.$$

Also, for $l=d+2,\ldots,2d+1-j$, given the points $P_1,\ldots,P_j,P_{d+2},\ldots,P_{l-1}$ we have

$$\mathrm{Prob}\,(h_l\leq y_l\leq h_l+dh_l)\leq d(c_d\cdot h_l^{2d-l-j+3})\leq c'_d\cdot h_l^{2d-l-j+2}dh_l\,.$$

To satisfy (7), given the points $P_1, P_2, \ldots, P_d$, the point P_{d+1} must lie in a box of volume at most

$$C_d' \cdot \frac{\beta}{\prod_{k=2}^{d} g_k} \,,$$

where $C_d' > 0$ is a constant. Similarly, if the points $P_1, \ldots, P_j, P_{d+2}, \ldots, P_{2d+1-j}$ are already fixed, to satisfy (8), the point P_{2d+2-j} must lie in a box of volume at most

$$C_d' \cdot \frac{\beta}{\prod_{k=2}^{j} g_k \cdot \prod_{l=d+2}^{2d+1-j} h_l} \,.$$

We infer for some constant $C_d > 0$:

$$\begin{aligned}
&\operatorname{Prob}(\mathcal{F}_{E,E'}) \\
&\quad = \operatorname{Prob}\Big(\{E, E'\} \text{ is a } (2,j)\text{-cycle in } \mathcal{G}(\beta) \mid \forall k \neq l : \operatorname{dist}(P_k, P_l) \geq D_0\Big) \\
&\quad \leq C_d \cdot \int_{D_0}^{\sqrt{d}} \cdots \int_{D_0}^{\sqrt{d}} \left(\prod_{k=2}^{d} g_k^{d+1-k} \right) \cdot \left(\prod_{l=d+2}^{2d+1-j} h_l^{2d-l-j+2} \right) \cdot \\
&\qquad \cdot \frac{\beta^2}{(\prod_{k=2}^{d} g_k) \cdot (\prod_{k=2}^{j} g_k) \cdot (\prod_{l=d+2}^{2d+1-j} h_l)} \cdot dh_{2d+1-j} \cdots dh_{d+2}\, dg_d \cdots dg_2 \\
&\quad = C_d \cdot \beta^2 \cdot \int_{D_0}^{\sqrt{d}} \cdots \int_{D_0}^{\sqrt{d}} \left(\prod_{k=2}^{j} g_k^{d-1-k} \right) \cdot \left(\prod_{k=j+1}^{d} g_k^{d-k} \right) \cdot \\
&\qquad \cdot \left(\prod_{l=d+2}^{2d+1-j} h_l^{2d-l-j+1} \right) dh_{2d+1-j} \cdots dh_{d+2}\, dg_d \cdots dg_2 \,.
\end{aligned}$$

For nonnegative exponents the terms g_k^{d-1-k} and $h_l^{2d-l-j+1}$ contribute with respect to the integration at most a constant factor dependent on d only. Only in the case $k = j = d$ the exponent $(d-1-k)$ of $g_k = g_d$ is negative. Hence, for $j = 2, \ldots, d-1$, we have for some constant $C_d^* > 0$

$$\operatorname{Prob}(\mathcal{F}_{E,E'}) \leq C_d^* \cdot \beta^2 \,, \tag{9}$$

while for $j = d$, and here we use the assumption $D_0 = n^{-2/(d-1)}$, we obtain for some constants $C_d', C_d'', C_d^{**} > 0$:

$$\begin{aligned}
\operatorname{Prob}(\mathcal{F}_{E,E'}) &\leq C_d' \cdot \beta^2 \cdot \int_{D_0}^{\sqrt{d}} \frac{1}{g_d} dg_d \leq C_d'' \cdot \beta^2 \cdot \log(1/D_0) \\
&\leq C_d^{**} \cdot \beta^2 \cdot \log n \,.
\end{aligned} \tag{10}$$

We can choose $(2d+2-j)$ points from $n^{1+\varepsilon}$ points in $\binom{n^{1+\varepsilon}}{2d+2-j}$ ways. Taking into account the number $\binom{2d-j+2}{d+1} \cdot \binom{d+1}{j}$ of possibilities to form a $(2,j)$-cycle, we conclude with (9) for $j=2,\ldots,d-1$, that the conditional expected numbers $E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \neq l : \text{dist}\,(P_k,P_l) \geq D_0)$ of $(2,j)$-cycles in $\mathcal{G}(\beta)$ satisfy for constants $c_j(d), c(d) > 0$:

$$\begin{aligned} &E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \neq l : \text{dist}\,(P_k,P_l) \geq D_0) \\ &\quad \leq \binom{2d-j+2}{d+1} \cdot \binom{d+1}{j} \cdot C_d^* \cdot \beta^2 \cdot \binom{n^{1+\varepsilon}}{2d+2-j} \\ &\quad \leq c_j(d) \cdot \beta^2 \cdot n^{(1+\varepsilon)(2d+2-j)}\,, \end{aligned}$$

and for $j=d$ we have by (10) that

$$\begin{aligned} &E(s_{2,d}(\mathcal{G}(\beta)) \mid \forall k \neq l : \text{dist}\,(P_k,P_l) \geq D_0) \\ &\quad \leq \binom{2d-j+2}{d+1} \cdot \binom{d+1}{j} \cdot C_d^{**} \cdot \beta^2 \cdot \log n \cdot \binom{n^{1+\varepsilon}}{d+2} \\ &\quad \leq c(d) \cdot \beta^2 \cdot n^{(1+\varepsilon)(d+2)} \cdot \log n\,. \end{aligned}$$

■

Now we set

$$\beta := \frac{\log n}{n^d}\,. \tag{11}$$

Lemma 2.6. *For fixed ε with $0 < \varepsilon < 1/(2d)$, there exists a hypergraph $\mathcal{G}(\beta) = (V, \mathcal{E})$ which satisfies:*

$$\begin{aligned} |V| &= n^{1+\varepsilon} \\ |\mathcal{E}| &\leq 2 \cdot C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)} \\ s_{2,j}(\mathcal{G}(\beta)) &\leq n \qquad \textit{for } j = 2, \ldots, d\,. \end{aligned}$$

Proof. We will show that the event $\mathcal{F} =$ "$(|\mathcal{E}| \leq 2 \cdot C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)})$ and $(\forall k \neq l : \text{dist}\,(P_k,P_l) \geq D_0)$ and $(\forall j : s_{2,j}(\mathcal{G}(\beta)) \leq n)$" happens with positive probability for our random hypergraph $\mathcal{G}(\beta) = (V, \mathcal{E})$.

The complementary event of $\mathcal{F}$ is $\overline{\mathcal{F}} =$ "$(|\mathcal{E}| > 2 \cdot C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)})$ or $(\exists k \neq l : \text{dist}\,(P_k,P_l) < D_0)$ or $(\exists j : [s_{2,j}(\mathcal{G}(\beta)) > n$ and $\forall k \neq l : \text{dist}\,(P_k,P_l) \geq D_0])$".

Using (1), (4) and Markov's inequality, i.e., $\text{Prob}\,(X \geq \alpha) \leq E(X)/\alpha$ for every real $\alpha > 0$ and every nonnegative random variable X, we infer

$$\begin{aligned} &\text{Prob}\,(\overline{\mathcal{F}}) \\ &\quad \leq \text{Prob}\left(|\mathcal{E}| > 2 \cdot C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)}\right) + \text{Prob}\left(\exists k \neq l : \text{dist}\,(P_k,P_l) < D_0\right) + \end{aligned}$$

$$+\,\mathrm{Prob}\Big(\exists\, j : [s_{2,j}(\mathcal{G}(\beta)) > n \text{ and } \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0]\Big)$$

$$\leq \frac{1}{2} + o(1) + \sum_{j=2}^{d} \mathrm{Prob}\Big(s_{2,j}(\mathcal{G}(\beta)) > n \text{ and } \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0\Big)$$

$$= \frac{1}{2} + o(1) + \sum_{j=2}^{d} \mathrm{Prob}\Big(\forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0\Big) \cdot$$

$$\cdot\, \mathrm{Prob}\Big(s_{2,j}(\mathcal{G}(\beta)) > n \mid \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0\Big)$$

$$= \frac{1}{2} + o(1) + (1 - o(1)) \cdot$$

$$\cdot \sum_{j=2}^{d} \mathrm{Prob}\Big(s_{2,j}(\mathcal{G}(\beta)) > n \mid \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0\Big)$$

$$\leq \frac{1}{2} + o(1) + (1 - o(1)) \cdot \sum_{j=2}^{d} \frac{E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0)}{n}\,. \tag{12}$$

For $j = 2, \ldots, d-1$ we have by (5) and (11) for $\varepsilon < 1/(2d)$:

$$\begin{aligned} &\frac{E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0)}{n} \\ &\quad \leq \frac{c_j(d) \cdot \beta^2 \cdot n^{(1+\varepsilon)(2d+2-j)}}{n} \\ &\quad = c_j(d) \cdot (\log n)^2 \cdot n^{1-j+\varepsilon(2d+2-j)} \\ &\quad = o(1)\,, \end{aligned}$$

and for $j = d$ and $\varepsilon < (d-1)/(d+2)$ we have by (6) and (11):

$$\begin{aligned} &\frac{E(s_{2,j}(\mathcal{G}(\beta)) \mid \forall k \neq l : \mathrm{dist}\,(P_k, P_l) \geq D_0)}{n} \\ &\quad \leq \frac{c(d) \cdot \beta^2 \cdot n^{(1+\varepsilon)(d+2)} \cdot \log n}{n} \\ &\quad = c(d) \cdot (\log n)^3 \cdot n^{-d+1+\varepsilon(d+2)} \\ &\quad = o(1)\,. \end{aligned}$$

We conclude with (12) that $\mathrm{Prob}\,(\overline{\mathcal{F}}) \leq 1/2 + o(1)$ and hence $\mathrm{Prob}\,(\mathcal{F}) > 0$ for $0 < \varepsilon < 1/(2d)$. Thus there exists a desired hypergraph $\mathcal{G}(\beta) = (V, \mathcal{E})$. ∎

We take the $(d+1)$-uniform hypergraph $\mathcal{G}(\beta)$ with $0 < \varepsilon < 1/(2d)$, which exists by Lemma 2.6, and we remove one vertex from each $(2,j)$-cycle, $j = 2, 3, \ldots, d$. We obtain an induced subhypergraph $\mathcal{G}_1(\beta) = (V_1, \mathcal{E}_1)$ of $\mathcal{G}(\beta) = (V, \mathcal{E})$ with $|V_1| = (1 - o(1)) \cdot n^{1+\varepsilon}$ vertices and $|\mathcal{E}_1| \leq 2 \cdot C_d \cdot \beta \cdot n^{(1+\varepsilon)(d+1)}$

and without any 2-cycles. Hence, $\mathcal{G}_1(\beta)$ has average degree at most $t^d = 2\cdot C_d\cdot(1+o(1))\cdot(d+1)\cdot\beta\cdot n^{(1+\varepsilon)d}$. Set $c_d^{**} := (2\cdot C_d\cdot(1+o(1))\cdot(d+1))^{1/d}$.

We apply Theorem 2.2 to the $(d+1)$-uniform subhypergraph $\mathcal{G}_1(\beta) = (V_1, \mathcal{E}_1)$ and by the choice of β in (11) the independence number $\alpha(\mathcal{G}_1(\beta))$ satisfies for suitable constants $c_d', c_d^* > 0$:

$$\begin{aligned}\alpha(\mathcal{G}(\beta)) \geq \alpha(\mathcal{G}_1(\beta)) &\geq c_{d+1}\cdot\frac{(1-o(1))\cdot n^{1+\varepsilon}}{c_d^{**}\cdot\beta^{1/d}\cdot n^{1+\varepsilon}}\cdot\left(\log(c_d^{**}\cdot\beta^{1/d}\cdot n^{1+\varepsilon})\right)^{1/d}\\ &\geq c_d'\cdot\frac{n}{(\log n)^{1/d}}\cdot(\log n^{\varepsilon})^{1/d}\\ &\geq c_d^*\cdot\frac{n}{(\log n)^{1/d}}\cdot(\log n)^{1/d}\\ &\geq c_d^*\cdot n\,.\end{aligned}$$

Thus, among the $n^{1+\varepsilon}$ points in $[0,1]^d$ there is a subset of $c_d^*\cdot n$ points, such that each simplex spanned by any $(d+1)$ of these $c_d^*\cdot n$ points has volume at least $\beta = \log n/n^d$. By adapting constant factors, i.e., choosing $\beta = c\cdot\log n/n^d$ for a suitable constant $c>0$, there exist n points in $[0,1]^d$ such that the volume of every simplex spanned by any $(d+1)$ of these n points is at least $\Omega(\log n/n^d)$. This finishes the proof of Theorem 1.1. ∎

3. Concluding Remarks

We showed by a probabilistic argument the existence of a configuration of n points in the d-dimensional unit cube $[0,1]^d$ such that the volume of every simplex formed by any $(d+1)$ of these points is at least $\Omega(\log n/n^d)$. Although there is an algorithmic version of Theorem 2.2 available, see [8] and [4], it seems to be difficult and involved, to turn our arguments into a *deterministic polynomial time algorithm.* For the 2-dimensional case we succeeded in doing so by using a sufficiently fine grid [5], and very recently also for the case $d=3$, see [11]. Moreover, it would also be interesting to investigate upper bounds for the d-dimensional version of Heilbronn's problem.

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